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The Second-Order Equation from the (1/2, 0) $\oplus$ (0, 1/2) Representation of the Poincaré Group

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On the basis of first principles we derive the Barut–Wilson–Fushchich second-order equation in the (1/2, 0) $\oplus$ (0, 1/2) representation. Then we discuss the possibility of describing various mass and spin states in such a framework.

A few decades ago Barut and collaborators proposed the use of the four-dimensional representation of the $O(4, 2)$ group in order to solve the problem of lepton mass splitting (Barut *et al.*, 1969, 1970; Wilson, 1974; Barut, 1978, 1979). Similar research has been produced by Fushchich and collaborators (Fushchich and Grishchenko, 1970; Fushchich and Nikitin, 1973, 1978).

The most general conserved current that is linear in the generators of the four-dimensional representation of the group $O(4, 2)$ was given as (Barut *et al.*, 1969, 1970; Wilson, 1974)

$$j_\mu = \alpha_1\gamma_\mu + \alpha_2 P_\mu + \alpha_3\sigma_{\mu\nu}q^\nu \tag{1}$$

where $P_\mu = p_{1\mu} + p_{2\mu}$ is the total momentum, $q_\mu = p_{1\mu} - p_{2\mu}$ is the momentum transfer, and α_i are real, constant coefficients which may depend on the internal degrees of freedom of leptons. The Lagrangian formalism and the secondary quantization scheme are given by Wilson (1974). Barut (1978, 1979) derived the mass spectrum of leptons after taking into account the additional postulate that one has to fix the value of the anomalous magnetic moment of the particle by its *classical* value $g = 2(2\alpha/3)$, where α is the fine structure constant.

Recently, we have approached the introduction of similar constructs in the $(j, 0) \oplus (0, j)$ representation from a very different standpoint (Dvoeglazov,

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1997a, b, d).[2] Starting from the explicit form of the spinors which are eigenspinors of the helicity operator (Varshalovich *et al.,* 1988), one can find *two nonequivalent* relations between zero-momentum 2-spinors. The first form of the Ryder–Burgard relation[3] is given in Ahluwalia (1996) (h is a quantum number for the helicity; in general, see the cited papers for the notation):

$$[\phi_L^h(p^\mu)]^* = e^{-2i\vartheta_h}\, \Xi_{[1/2]}\, \phi_L^h(p^\mu) \tag{2}$$

The second form is given in Dvoeglazov (1995):

$$[\phi_L^h(p^\mu)]^* = (-1)^{1/2-h}\, e^{-i(\vartheta_1+\vartheta_2)}\, \Theta_{[1/2]}\, \phi_L^{-h}(p^\mu) \tag{3}$$

The matrices are defined in the $(1/2, 0) \oplus (0, 1/2)$ representation as follows:

$$\Theta_{[1/2]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Xi_{[1/2]} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{4}$$

Here ϕ is the azimuthal angle related to $\mathbf{p} \to \mathbf{0}$.

In this paper we use the generalized Ryder–Burgard relation

$$\phi_L^h(p^\mu) = a(-1)^{1/2-h} e^{i(\vartheta_1+\vartheta_2)}\, \Theta_{[1/2]}[\phi_L^{-h}(p^\mu)]^* + b e^{2i\vartheta_h}\, \Xi_{[1/2]}^{-1}[\phi_L^h(p^\mu)]^* \tag{5}$$

with the *real* constants a and b being arbitrary at this stage. The relevant relations for ϕ_R are obtained after taking into account that

$$\phi_L^{\uparrow}(p^\mu) = -\Theta_{[1/2]}[\phi_R^{\downarrow}(p^\mu)]^*, \qquad \phi_L^{\downarrow}(p^\mu) = +\Theta_{[1/2]}\,[\phi_R^{\uparrow}(p^\mu)]^* \tag{6a}$$

$$\phi_R^{\uparrow}(p^\mu) = -\Theta_{[1/2]}[\phi_L^{\downarrow}(p^\mu)]^*, \qquad \phi_R^{\downarrow}(p^\mu) = +\Theta_{[1/2]}\,[\phi_L^{\uparrow}(p^\mu)]^* \tag{6b}$$

which are easily derived by considering the explicit form of 4-spinors, (e.g., Ryder, 1985).[4] Next we apply the procedure outlined in Ahluwalia (1996,

[2] In Dvoeglazov (1997a) I used the Tucker–Hammer (1971) modification of Weinberg's (1964a, b) equations.

[3] This name was introduced by Ahluwalia *et al.* (1993) in considering the $(1, 0) \oplus (0, 1)$ representation. If one uses $\phi_R(p^\mu) = \pm\phi_L(p^\mu)$ (cf. also Faustov, 1971; Ryder, 1985), after application of the Wigner rules for the boosts of the 2-spinors to the momentum p^μ one immediately arrives at the Bargmann–Wightman–Wigner type quantum field theory (Wigner, 1965; cf. also Gel'fand and Tsetlin, 1957; Sokolik, 1958) in this representation. [Note that equation (22a) of Faustov (1971) reads $Bu_\lambda(0) = u_\lambda(0)$, where $u_\lambda(\overline{p})$ is a $2(2j + 1)$-spinor and $B^2 = 1$ is an $2(2j + 1) \times 2(2j + 1)$ matrix with the above-mentioned property.]

[4] In fact, we have a certain amount of room in the definitions of the right spinors due to the arbitrariness of the phase factors at this stage. But if one chooses the spinorial basis as in Ryder (1985), one can find the relevant spinors in any frame after the application of the Wigner rules

$$\phi_{R,L}(p^\mu) = \exp(\pm\sigma \cdot \hat{\varphi}/2)\phi_{R,L}(p^\mu) \tag{7}$$

for right (left) spinors); $\cosh(\varphi) = E/m$, $\sinh(\varphi) = |\mathbf{p}|/m$, and $\hat{\varphi} = \mathbf{p}/|\mathbf{p}|$ is the unit vector. In subsequent papers we shall consider different choices of the phase factors between left and right spinors in detail.

footnote 1). Namely,

$$
\begin{aligned}
\phi_L^h(p^\mu) &= \Lambda_L(p^\mu \leftarrow \mathring{p}^\mu))\phi_L^h(\mathring{p}^\mu) \\
&= \Lambda_L(p^\mu \leftarrow \mathring{p}^\mu)\ \{a(-1)^{1/2-h}e^{i(\vartheta_1+\vartheta_2)}\Theta_{[1/2]}[\phi_L^{-h}(\mathring{p}^\mu)]^* \\
&\quad + be^{2i\vartheta_h}\Xi_{[1/2]}^{-1}[\phi_L^h(\mathring{p}^\mu)]^*\} \\
&= -ae^{i(\vartheta_1+\vartheta_2)}\Lambda_L(p^\mu \leftarrow \mathring{p}^\mu)\Lambda_R^{-1}(p^\mu \leftarrow \mathring{p}^\mu)\phi_R^h(p^\mu) \\
&\quad + be^{2i\vartheta_h}(-1)^{1/2+h}\Theta_{[1/2]}\Xi_{[1/2]}\phi_R^{-h}(p^\mu)
\end{aligned}
\tag{8}
$$

As a consequence of equations (6a), (6b), (7), and (8), after the choice of the phase factors (e.g., $\vartheta_1 = 0$, $\vartheta_2 = \pi$), one has

$$
\phi_L^h(p^\mu) = a\,\frac{p_0 - \boldsymbol{\sigma}\cdot\mathbf{p}}{m}\,\phi_R^h(p^\mu) + b(-1)^{1/2+h}\Theta_{[1/2]}\Xi_{[1/2]}\phi_R^{-h}(p^\mu) \tag{9}
$$

$$
\phi_R^h(p^\mu) = a\,\frac{p_0 + \boldsymbol{\sigma}\cdot\mathbf{p}}{m}\,\phi_L^h(p^\mu) + b(-1)^{1/2+h}\Theta_{[1/2]}\Xi_{[1/2]}\phi_L^{-h}(p^\mu) \tag{10}
$$

Thus, the momentum-space Dirac equation is generalized:

$$
\left(a\,\frac{\hat{p}}{m} - \mathbb{1}\right)u_h(p^\mu) + ib(-1)^{1/2-h}\gamma^5\mathcal{C}u^*_{-h}(p^\mu) = 0 \tag{11}
$$

where

$$
\mathcal{C} = \begin{pmatrix} 0 & i\Theta_{[1/2]} \\ -i\Theta_{[1/2]} & 0 \end{pmatrix} \tag{12}
$$

is a 4×4 matrix which enters in the definition of the charge conjugation operation. The counterpart in the coordinate space is

$$
\left[a\,\frac{i\gamma^\mu\partial_\mu}{m} + b\mathcal{C}\mathcal{K} - \mathbb{1}\right]\Psi(x^\mu) = 0 \tag{13}
$$

provided that in the operator formulation the creation (annihilation) operators are connected by

$$
b_\downarrow(p^\mu) = -ia_\uparrow(p^\mu), \qquad b_\uparrow(p^\mu) = +ia_\downarrow(p^\mu) \tag{14}
$$

We denote by $\mathcal{K}$ the operation of complex conjugation, and note that it acts as the Hermitian conjugation on the creation (annihilation) operators in q-number theories.

On the other hand, we have the possibility of dividing the Dirac function into real and imaginary parts (Majorana, 1937). The transformation from the

Weyl representation of the γ^μ matrices is made by means of the unitary 4×4 matrix

$$U = \frac{1}{2}\begin{pmatrix} \mathbb{1} - i\Theta_{[1/2]} & \mathbb{1} + i\Theta_{[1/2]} \\ -\mathbb{1} - i\Theta_{[1/2]} & \mathbb{1} - i\Theta_{[1/2]} \end{pmatrix},$$
$$U^\dagger = \frac{1}{2}\begin{pmatrix} \mathbb{1} - i\Theta_{[1/2]} & -\mathbb{1} - i\Theta_{[1/2]} \\ \mathbb{1} + i\Theta_{[1/2]} & \mathbb{1} - i\Theta_{[1/2]} \end{pmatrix} \tag{15}$$

Similar transformations can also be applied to the Weinberg (or Weinberg–Hammer–Tucker) spin-1 equation in order to divide it into real and imaginary parts (Dvoeglazov, 1997c). Presumably, this is a general property of all fundamental wave equations.

Let us write the generalized coordinate-space Dirac equation (13) in the Majorana representation:

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} - b\mathcal{K} - \mathbb{1}\right]\Psi(x^\mu) = 0 \tag{16}$$

We used that $U\mathcal{C}\mathcal{K}U^{-1} = -\mathcal{K}$. Following Majorana, we note that the γ^μ matrices become *pure imaginary* and the Dirac function can be divided into $\Psi \equiv \Psi_1 + i\Psi_2$. Hence, we have a set of two *real* equations:

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} - b - \mathbb{1}\right]\Psi_1(x^\mu) = 0 \tag{17a}$$

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} + b - \mathbb{1}\right]\Psi_2(x^\mu) = 0 \tag{17b}$$

Adding and subtracting the obtained equations, we arrive at the set ($\phi = \Psi_1 + \Psi_2$ and $\chi = \Psi_1 - \Psi_2$)

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} - \mathbb{1}\right]\phi - b\chi = 0 \tag{18a}$$

$$\left[a\frac{i\gamma^\mu\partial_\mu}{m} - \mathbb{1}\right]\chi - b\phi = 0 \tag{18b}$$

which after multiplication by $b \neq 0$ yield the same second-order equations for ϕ and χ:

$$\left[2a\frac{i\gamma^\mu\partial_\mu}{m} + a^2\frac{\partial^\mu\partial_\mu}{m^2} + b^2 - \mathbb{1}\right]\begin{cases}\phi(x^\mu)\\ \chi(x^\mu)\end{cases} = 0 \tag{19}$$

With the identifications $a/(2m) \to \alpha_2$ and $[(1 - b^2)/2a]m \to \kappa$, we obtain equation (6) of Barut (1978). This is the equation which Barut used to obtain the mass difference between a muon and an electron.

On using a similar technique and considering different *chirality* subspaces as independent, one obtains the Fushchich equation (Fushchich and Nikitin, 1978). It has the interaction with the 4-vector potential which is the same as the $j = 0$ Sakata–Taketani particle. This problem is closely related to the recent discussion of the Dirac equation with *two* mass parameters (Raspini, 1996). It seems to me that the theoretical difference (compared with the first-order Dirac equation) which we obtain after switched-on interactions is caused by the induced asymmetry between evolutions toward and backward in time (or, perhaps equivalently, between different chirality/helicity subspaces).

In conclusion, we can now assert that the Barut proposal was not some puzzling coincidence. In fact, the second-order equation can be derived on the basis of a minimal number of postulates (Lorentz invariance and relations between 2-spinors in the zero-momentum frame) and it is a natural consequence of the general structure of the $(j, 0) \oplus (0, j)$ representation.

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